Examples for linear algebra

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1. Solve using the Gaussian elimination,

$$3x_1 + 8x_2 = 53$$
$$6x_1 + 2x_2 = 50$$

Solution. Write the equations in an augmented matrix,

$$[\,A \quad B\,] = \begin{bmatrix} 3 & 8 & 53 \\ 6 & 2 & 50 \end{bmatrix}$$

Row 1 times $\frac{1}{3}$,

$$\begin{bmatrix} 1 & \frac{8}{3} & \frac{53}{3} \\ 6 & 2 & 50 \end{bmatrix}$$

Row 2 subtracted by 6 times of Row 1,

$$\begin{bmatrix} 1 & \frac{8}{3} & \frac{53}{3} \\ 0 & -14 & -56 \end{bmatrix}$$

Row 2 times $-\frac{1}{14}$,

$$\begin{bmatrix} 1 & \frac{8}{3} & \frac{53}{3} \\ 0 & 1 & 4 \end{bmatrix}$$

Row 1 subtracted by $\frac{8}{3}$ times of Row 2,

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$$

Therefore $\bar{x}_1 = 7$ and $\bar{x}_2 = 4$.

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2. Let

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

Find all the eigenvalues and a basis of each eigenspace. Can A be diagonalised? Why?

Solution. Form the characteristic matrix $tI - A$ and find its determinant to obtain the characteristic polynomial $\Delta(t)$ of A;

$$\Delta(t) = |\,tI - A\,| = \begin{vmatrix} t-1 & 3 & -3 \\ -3 & t+5 & -3 \\ -6 & 6 & t-4 \end{vmatrix} = (t+2)^2(t-4)$$

The roots of $\Delta(t)$ are -2 and 4, hence the eigenvalues of A are -2 and 4.

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Next, find a basis of the eigenspace of the eigenvalue -2. Substitute $t = -2$ into the characteristic matrix $tI - A$, thus obtain the homogeneous system

$$\begin{pmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In other words,

$$\begin{cases} -3x + 3y - 3z = 0 \\ -3x + 3y - 3z = 0 \\ -6x + 6y - 6z = 0 \end{cases}$$

that is $x - y + z = 0$. The system has two independent solutions, for example $x = 1$, $y = 1$, $z = 0$ and $x = 1$, $y = 0$, $z = -1$. Therefore $u = (1, 1, 0)$ and $v = (1, 0, -1)$ are independent eigenvectors which generate the eigenspace of the eigenvalue -2. In other words, u and v form a basis of the eigenspace of -2. This means that every eigenvector belonging to -2 is a linear combination of u and v.

Similarly, find a basis of the eigenspace of the eigenvalue 4. Substitute $t = 4$ into the characteristic matrix $tI - A$ to obtain the homogeneous system

$$\begin{pmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is the same as writing

$$\begin{cases} 3x + 3y - 3z = 0 \\ -3x + 9y - 3z = 0 \\ -6x + 6y = 0 \end{cases}$$

which can be reduced to

$$\begin{cases} x + y - z = 0 \\ 2y - z = 0 \end{cases}$$

Since the number of independent equations is less than the number of variables by one, this system has only one free variable. Therefore any particular non-zero solution, for example $x = 1$, $y = 1$, $z = 2$ generates its solution space. Hence $w = (1, 1, 2)$ is an eigenvector which generates, and so form a basis of the eigenspace of the eigenvalue 4.

Since A has three linearly independent eigenvectors, A is diagonalisable.

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Let P be the matrix the columns of which are the three independent eigenvectors,

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

Then

$$P^{-1}AP = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The diagonal elements of $P^{-1}AP$ are the eigenvalues of A corresponding to the columns of P.

Bibliography

Seymour Lipschutz. *Theory and problems of Linear algebra.* Schaum's Outline Series, 1987(1968)